Journal of Engineering Mathematics **25**: 287–299, 1991.

On the derivation of first integrals for similarity solutions

JAMES M. HILL and DESMOND L. HILL[1]

Department of Mathematics, The University of Wollongong, Wollongong, N.S.W. 2500, Australia;
[1]*present address, Department of Mathematics, University of Western Australia, Nedlands, Western Australia*

Received 1 June 1990; accepted 4 September 1990

Abstract. In two recent papers the authors have obtained a number of first integrals for similarity solutions of nonlinear diffusion and of general high-order nonlinear evolution equations. Such integrals exist only for special parameter values and are obtained via integration of the ordinary differential equation, which results when the functional form of the solution is substituted into the governing partial differential equation. In this paper we show that these special parameter values also occur in a natural way when we utilize the first order partial differential equation instead of the explicit functional form and we ask under what conditions can a first integral with respect to either of the independent variables x or t be deduced. This simple procedure generates all previous results and presents the idea of similarity solutions in an entirely new light. That is, the significant features of similarity solutions for partial differential equations are not necessarily the explicit functional form and subsequent reduction to an ordinary differential equation but rather that the solutions sort are common to two partial differential equations. The process is illustrated with reference to an extensive number of examples including nonlinear diffusion, general diffusion equations containing a number of parameters and high-order nonlinear evolution equations. In addition a new exact solution for nonlinear diffusion is obtained which is illustrated graphically.

1. Introduction

The basic idea of any similarity solution is that an assumed functional form of the solution enables a partial differential equation to be reduced to an ordinary differential equation (or to a partial differential equation of lower order). In two recent papers Hill [6] and Hill and Hill [7] the authors have deduced first integrals for stretching similarity solutions of the nonlinear diffusion equation and generally for a class of high-order nonlinear evolution equations. The purpose of this paper is to present an alternative derivation of these first integrals utilizing the first order partial differential equation rather than the explicit functional form of the similarity solution. This approach presents similarity solutions in an entirely new light and by equal utilization of both the original partial differential equation and the first order partial differential equation we are able to deduce previous integrals virtually immediately without reference to the underlying ordinary differential equation.

We first summarize the results presented in [6] for nonlinear diffusion with power law diffusivity and with governing partial differential equation,

$$\frac{\partial c}{\partial t} = \frac{\partial}{\partial x}\left(c^m \frac{\partial c}{\partial x}\right). \tag{1.1}$$

As noted in [6], this equation remains invariant under the one-parameter group

$$x_1 = e^{(\lambda+1)\varepsilon}x , \qquad t_1 = e^{2\varepsilon}t , \qquad c_1 = e^{2\lambda\varepsilon/m}c , \tag{1.2}$$

where λ denotes an arbitrary constant and accordingly admits similarity solutions of the form

$$c(x, t) = x^{2\lambda/m(1+\lambda)}\phi(\xi), \qquad \xi = \frac{x^{1/(1+\lambda)}}{t^{1/2}}, \tag{1.3}$$

assuming that $\lambda \neq -1$. Now on substitution of (1.3) into the partial differential equation (1.1), it happens that the resulting second order nonlinear ordinary differential equation for $\phi(\xi)$ admits a first integral for the two values of λ, namely

$$\lambda = \frac{-m}{(m+2)}, \qquad \lambda = \frac{-m}{(m+1)}, \tag{1.4}$$

which curiously enough turn out to be precisely the values of λ applicable to the well known point source solution and the so-called dipole solution respectively (see [6]). In this paper we present a simple interpretation as to why these particular values of λ exhibit a first integral and we do so making use of the first order partial differential equation

$$(\lambda+1)x\frac{\partial c}{\partial x} + 2t\frac{\partial c}{\partial t} = \frac{2\lambda}{m}c, \tag{1.5}$$

rather than the equivalent similarity functional form (1.3).

It turns out that the values of λ (1.4) are those which enable equation (2.1) to be integrated directly with respect to the independent variables x and t and these details are presented in the following section. In the subsequent section we show that the procedure is effective in producing other first integrals for nonlinear diffusion which are given in [6]. In Section 4 we demonstrate that the approach can also be used for general diffusion equations of the form

$$\frac{\partial c}{\partial t} = x^l \frac{\partial}{\partial x}\left\{c^m x^n \left|\frac{\partial c}{\partial x}\right|^p \frac{\partial c}{\partial x}\right\}, \tag{1.6}$$

which includes an equation recently proposed by Grundy [3] as a model for rock blasting. Integrals for (1.6) with p zero have been derived in Hill [6] and various special solutions for l, m and n all zero have been recently given by the authors (Hill and Hill [5]). Integrals for other special cases of (1.6) can also be found in Atkinson and Jones [1]. In Section 5 recent results given in Hill and Hill [7] for high-order nonlinear evolution equations are deduced by the procedure described here. Finally in Section 6 we utilize this approach to derive a new exact solution for the nonlinear diffusion equation (1.1) valid for $m = 1$, which is not included in the exact solutions given in Hill and Hill [4]. Finally in this section, we note the recent interesting article by King [8] relating to other exact solutions of the nonlinear diffusion equation.

2. Alternative derivation of integrals for nonlinear diffusion

In this section we illustrate the process of deducing simple integrals with reference to the similarity solution (1.3) of the nonlinear diffusion equation (1.1). On eliminating $\partial c/\partial t$ between (1.1) and (1.5) we obtain

$$(\lambda+1)x\frac{\partial c}{\partial x} - \frac{2\lambda}{m}c + 2t\frac{\partial}{\partial x}\left(c^m \frac{\partial c}{\partial x}\right) = 0, \tag{2.1}$$

and we ask the question: "Under what conditions can be integrate this equation with respect to the variable x?" Clearly if $(\lambda + 1)$ equals $-2\lambda/m$, that is λ is given by $(1.4)_1$, then (2.1) becomes

$$\frac{\partial}{\partial x}\left\{\frac{2xc}{(m+2)} + 2tc^m \frac{\partial c}{\partial x}\right\} = 0\,, \tag{2.2}$$

and therefore we have

$$tc^m \frac{\partial c}{\partial x} + \frac{xc}{(m+2)} = f_1(t)\,, \tag{2.3}$$

where $f_1(t)$ denotes a function of t only. If we now require (2.3) to be also invariant under (1.2) then we find that $f_1(t_1) \equiv f_1(t)$ and therefore $f_1(t)$ is at most a constant C_1. On substituting the similarity from (1.3) into equation (2.3) we find that the resulting integral is precisely that in Hill [6] obtained via an integration of the second order nonlinear ordinary differential equation for $\phi(\xi)$.

Similarly, if we multiply (2.1) by x then the resulting equation can be integrated immediately with respect to x provided $(\lambda + 1)$ equals $-\lambda/m$, in which case λ is given by $(1.4)_2$ and (2.1) becomes

$$\frac{\partial}{\partial x}\left\{\frac{x^2 c}{(m+1)} + 2t\left[xc^m \frac{\partial c}{\partial x} - \frac{c^{m+1}}{(m+1)}\right]\right\} = 0\,, \tag{2.4}$$

and therefore we have

$$\frac{x^2 c}{(m+1)} + 2t\left[xc^m \frac{\partial c}{\partial x} - \frac{c^{m+1}}{(m+1)}\right] = f_2(t)\,, \tag{2.5}$$

where $f_2(t)$ denotes a function of t only. Again on requiring that (2.5) is also invariant under (1.2) we again deduce that $f_2(t)$ is at most a constant C_2 and substitution of (1.3) into (2.5) again yields precisely the integral given in Hill [6].

Alternatively, we may use the first order partial differential equation (1.5) to eliminate $\partial c/\partial x$ from (1.1), thus

$$\frac{\partial c}{\partial t} + \frac{2}{(\lambda+1)}\frac{\partial}{\partial x}\left\{\frac{tc^m}{x}\frac{\partial c}{\partial t} - \frac{\lambda c^{m+1}}{mx}\right\} = 0\,, \tag{2.6}$$

and on this occasion we may ask the question: "Under what conditions can we integrate this equation with respect to the variable t?" In this case, if $(m+1)^{-1}$ equals $-\lambda/m$, that is λ is given by $(1.4)_2$ then (2.6) becomes

$$\frac{\partial}{\partial t}\left\{c + 2\frac{\partial}{\partial x}\left[\frac{tc^{m+1}}{x}\right]\right\} = 0\,, \tag{2.7}$$

and therefore we have

$$c + 2\frac{\partial}{\partial x}\left[\frac{tc^{m+1}}{x}\right] = g(x)\,, \tag{2.8}$$

where $g(x)$ denotes a function of x only. If (2.8) is invariant under (1.2), we require that $g(x)$ transforms like c and therefore we have

$$g(x_1) = e^{-2\varepsilon/(m+1)} g(x) , \tag{2.9}$$

from which we may deduce

$$g(x) = \frac{(m+1)C_2}{x^2} , \tag{2.10}$$

where C_2 denotes a constant, and this integral coincides with that obtained from (2.5).

Clearly, we have demonstrated that the integrals previously obtained via the ordinary differential equation for $\phi(\xi)$ can also be deduced from much simpler considerations. In subsequent sections we show that this interpretation also holds for other known first integrals and for general equations such as (1.6) it provides by far the simplest mechanism to deduce such results.

3. Other integrals for nonlinear diffusion

In this section we demonstrate that the process described in the previous section also applies to the other integrals deduced in Hill [6]. First the nonlinear diffusion equation with exponential diffusivity $D(c) = \alpha\, e^{\beta c}$, namely

$$\frac{\partial c}{\partial t} = \alpha \frac{\partial}{\partial x} \left\{ e^{\beta c} \frac{\partial c}{\partial x} \right\} , \tag{3.1}$$

where α and β are constants, remains invariant under the one-parameter group of transformations

$$x_1 = x + \frac{\varepsilon}{\gamma} , \qquad t_1 = e^{2\varepsilon} t , \qquad c_1 = c - \frac{2\varepsilon}{\beta} , \tag{3.2}$$

for arbitrary γ. This means that the functional form of the solution is obtained by solving the first order partial differential equation

$$2t \frac{\partial c}{\partial t} + \frac{1}{\gamma} \frac{\partial c}{\partial x} = -\frac{2}{\beta} , \tag{3.3}$$

and consequently, as noted in [6], has the form

$$e^{\beta c} = e^{-2\gamma x} \phi(\omega) , \quad \omega = e^{\gamma x}/t^{1/2} . \tag{3.4}$$

On eliminating $\partial c/\partial t$ from (3.1) and (3.3) we obtain an equation which readily integrates with respect to x to yield

$$\frac{c}{\gamma} + 2\alpha t\, e^{\beta c} \frac{\partial c}{\partial x} = -\frac{2x}{\beta} + f(t) , \tag{3.5}$$

which is also invariant under (3.2) provided that $f(t)$ is constant and the value $2\alpha\gamma C_1/\beta$

yields precisely the integral given in [6]. (We note that equations (4.5) and (4.6) of [6] contain a minor typographical error, namely that the constant β should not occur in these equations.) Similarly, if we eliminate $\partial c/\partial x$ from (3.1) and (3.3) and perform an integration with respect to t then we may deduce

$$\frac{c}{\gamma} + 2\alpha t\, e^{\beta c} \frac{\partial c}{\partial x} = g(x)\,, \tag{3.6}$$

and invariance under (3.2) requires that the function $g(x)$ satisfies

$$g(x_1) = g(x) - \frac{2\varepsilon}{\beta\gamma}\,, \tag{3.7}$$

from which we may deduce $g'(x) = -2/\beta$ and therefore (3.6) coincides with the integral obtained via the integration with respect to x. We notice that on this occasion there appears to be only the one integral.

Returning now to the nonlinear diffusion equation with power law diffusivity (1.1), as noted in [6] this equation has solutions

$$c(x, t) = x^{2/m}\phi(\xi)\,, \qquad \xi = x\, e^{-\alpha m t/2}\,, \tag{3.8}$$

for arbitrary α, which arises from invariance under the one-parameter group

$$x_1 = e^{\varepsilon} x\,, \qquad t_1 = t + \frac{2\varepsilon}{\alpha m}\,, \qquad c_1 = e^{2\varepsilon/m} c\,, \tag{3.9}$$

and the corresponding first order partial differential equation

$$x\frac{\partial c}{\partial x} + \frac{2}{\alpha m}\frac{\partial c}{\partial t} = \frac{2}{m} c\,. \tag{3.10}$$

On eliminating $\partial c/\partial t$ between (1.1) and (3.10) we may deduce

$$x\frac{\partial c}{\partial x} - \frac{2}{m} c + \frac{2}{\alpha m}\frac{\partial}{\partial x}\left(c^m \frac{\partial c}{\partial x}\right) = 0\,, \tag{3.11}$$

which evidently integrates with respect to x when $m = -2$ giving rise to the integral noted in Hill [6] and due originally to Grundy [2]. However, a new integral arises from (3.11) for $m = -1$ after multiplying the equation by x. We may readily deduce

$$x^2 c - \frac{2}{\alpha}\left(\frac{x}{c}\frac{\partial c}{\partial x} - \log c\right) = f(t)\,, \tag{3.12}$$

and invariance under (3.9) requires that $f(t_1) = f(t) - 4\varepsilon/\alpha$ and therefore $f(t) = 2t + C_2$ and altogether from (3.8) and (3.12) we have

$$c(x, t) = \frac{\phi(\xi)}{x^2}\,, \qquad \xi = x\, e^{\alpha t/2}\,, \tag{3.13}$$

with the new first integral

$$\phi - \frac{2}{\alpha}\left\{\frac{(\xi\phi' - 2\phi)}{\phi} - \log\phi + 2\log\xi\right\} = C_2\,, \tag{3.14}$$

which although it can be simplified, appears not to admit a further integration.

Finally in this section, we show that the first integral and the new solution given in [6] for the nonlinear diffusion equation (1.1) with index $m = -4/3$ also emerges from this procedure. As described in [6] the functional form of the similarity solution is

$$c(x, t) = \frac{\phi(\xi)}{[(x - x_1)^{1-\sigma}(x - x_2)^{1+\sigma}]^{3/2}}, \qquad \xi = \frac{1}{t^{1/2}} \left| \frac{x - x_1}{x - x_2} \right|^{\sigma}, \tag{3.15}$$

where x_1 and x_2 denote the two roots of the quadratic equation

$$\mu x^2 + (1 + \lambda)x + \kappa = 0, \tag{3.16}$$

and σ denotes $[(1 + \lambda)^2 - 4\mu\kappa]^{-1/2}$. This functional form arises by solving the first order partial differential equation

$$[\mu x^2 + (1 + \lambda)x + \kappa] \frac{\partial c}{\partial x} + 2t \frac{\partial c}{\partial t} = -\frac{3}{2} c(2\mu x + \lambda), \tag{3.17}$$

where λ, μ and κ are constants arising in the one-parameter group which leaves (1.1) with $m = -4/3$ invariant. In [6] by utilization of the functional from (3.15), the second order ordinary differential equation for $\phi(\xi)$ is shown to admit a first integral provided the constants λ, μ and κ are such that

$$(1 + \lambda)^2 - 4\mu\kappa = 9, \tag{3.18}$$

that is, provided $\sigma = \pm 1/3$ and the two values of σ giving essentially the same solution. In order to see why these values of σ are significant we need the identity

$$\frac{\partial}{\partial x} \{(x - x_1)^{1-\sigma}(x - x_2)^{1+\sigma}\}^{3/2} = \frac{3}{2\mu} (x - x_1)^{(1-3\sigma)/2}(x - x_2)^{(1+3\sigma)/2}(2\mu x + \lambda), \tag{3.19}$$

so that (3.17) becomes on eliminating $\partial c/\partial t$ by means of (1.1) with $m = -4/3$,

$$\frac{\partial}{\partial x} \{[(x - x_1)^{1-\sigma}(x - x_2)^{1+\sigma}]^{3/2} c\} + \frac{2t}{\mu} (x - x_1)^{(1-3\sigma)/2}(x - x_2)^{(1+3\sigma)/2} \frac{\partial}{\partial x} \left\{ c^{-4/3} \frac{\partial c}{\partial x} \right\} = 0. \tag{3.20}$$

This equation can be integrated with respect to x only if $(x - x_1)^{(1-3\sigma)/2}(x - x_2)^{(1+3\sigma)/2}$ is at most a linear function of x, which is only the case when $\sigma = \pm 1/3$. When $\sigma = 1/3$ equation (3.20) becomes

$$\frac{\partial}{\partial x} \{(x - x_1)(x - x_2)^2 c\} + \frac{2t}{\mu} (x - x_2) \frac{\partial}{\partial x} \left\{ c^{-4/3} \frac{\partial c}{\partial x} \right\} = 0, \tag{3.21}$$

which integrates to give

$$(x - x_1)(x - x_2)^2 c + \frac{2t}{\mu} \left\{ (x - x_2) c^{-4/3} \frac{\partial c}{\partial x} + 3c^{-1/3} \right\} = f(t), \tag{3.22}$$

which using the global form of the one-parameter group can be reconciled with the integral given in [6].

4. Integrals for general nonlinear diffusion

In this section we derive the integral (4.18) for the general equation (1.6) which contains four arbitrary constants l, m, n and p and includes a wide variety of different physical phenomena. For example this equation includes nonlinear diffusion in cylindrical and spherical regions, nonhomogeneous diffusivity and also applies to the flow of a non-Newtonian fluid in a porous medium (see for example Hill and Hill [5]). However, we first consider the special case of (1.6) with the parameter p zero which admits two first integrals. Thus we consider

$$\frac{\partial c}{\partial t} = x^l \frac{\partial}{\partial x}\left\{c^m x^n \frac{\partial c}{\partial x}\right\}, \tag{4.1}$$

which remains invariant under the one-parameter group

$$x_1 = \mathrm{e}^{(\lambda+1)\varepsilon} x, \qquad t_1 = \mathrm{e}^{2\varepsilon} t, \qquad c_1 = \mathrm{e}^{a\varepsilon} c, \tag{4.2}$$

with the constant a given by

$$a = \frac{2\lambda}{m} - \frac{(n+l)(\lambda+1)}{m}. \tag{4.3}$$

Accordingly the similarity solution $c(x, t)$ satisfies the first order partial differential equation, thus

$$(\lambda+1)x \frac{\partial c}{\partial x} + 2t \frac{\partial c}{\partial t} = ac. \tag{4.4}$$

Now on eliminating $\partial c/\partial t$ between (4.1) and (4.4) we have

$$x^{1-l} \frac{\partial c}{\partial x} - \frac{ax^{-l}}{(\lambda+1)} c + \frac{2t}{(\lambda+1)} \frac{\partial}{\partial x}\left\{c^m x^n \frac{\partial c}{\partial x}\right\} = 0, \tag{4.5}$$

and the condition that (4.5) admits a simple integral with respect to x is simply

$$1 - l = -a/(\lambda+1), \tag{4.6}$$

in which case we obtain

$$x^{1-l} c + \frac{2t}{(\lambda+1)} c^m x^n \frac{\partial c}{\partial x} = f(t). \tag{4.7}$$

From (4.2) and the condition (4.6) we may deduce that $f(t)$ is at most a constant.

Alternatively, if we use (4.4) to eliminate $\partial c/\partial x$ from (4.1) then we have

$$\frac{\partial c}{\partial t} + \frac{2x^l}{(\lambda+1)} \frac{\partial}{\partial x}\left\{\frac{x^{n-1}}{(m+1)}\left(t \frac{\partial c^{m+1}}{\partial t} - \frac{a(m+1)}{2} c^{m+1}\right)\right\} = 0, \tag{4.8}$$

which evidently admits an integration with respect to t provided

$$1 = -a(m+1)/2, \tag{4.9}$$

in which case we have

$$c + \frac{2tx^{l}}{(\lambda+1)(m+1)} \frac{\partial}{\partial x} \{x^{n-1}c^{m+1}\} = g(x) , \tag{4.10}$$

and from (4.2) and (4.9) we may deduce

$$g(x) = Cx^{-2/(m+1)(\lambda+1)} , \tag{4.11}$$

where C denotes an arbitrary constant. It is not difficult to show that the above conditions on the constant a give rise to precisely the λ values given in Hill [6] for which the similarity solution permits a first integral.

It is instructive to show that the λ value arising from (4.9) also arises as follows. In multiplying (4.5) by x^N and requiring that the result is an exact differential with respect to x, then for the first terms we clearly require

$$N + 1 - l = -a/(\lambda+1) , \tag{4.12}$$

and integration by parts is effective for the final term provided $N = 1 - n$ and (4.12) becomes

$$2 - (n + l) = -a/(\lambda+1) , \tag{4.13}$$

which turns out to yield the same λ value as (4.9) with the integral coinciding with (4.10) and (4.11).

For the more general equation (1.6) only the first of these approaches is effective. Equation (1.6) is invariant under the one-parameter group

$$x_1 = e^{(\lambda+1)\varepsilon}x , \qquad t_1 = e^{2\varepsilon}t , \qquad c_1 = e^{b\varepsilon}c , \tag{4.14}$$

with the constant b given by

$$b = \frac{2\lambda}{(m+p)} - \frac{(n+l-p)(\lambda+1)}{(m+p)} , \tag{4.15}$$

and from (1.6) and the appropriate first order partial differential equation we may deduce

$$x^{1-l}\frac{\partial c}{\partial x} - \frac{bx^{-l}}{(\lambda+1)}c + \frac{2t}{(\lambda+1)}\frac{\partial}{\partial x}\left\{c^m x^n \left|\frac{\partial c}{\partial x}\right|^p \frac{\partial c}{\partial x}\right\} = 0 . \tag{4.16}$$

The equation evidently permits an integration with respect to x provided

$$1 - l = -b/(\lambda+1) , \tag{4.17}$$

with first integral

$$x^{1-l}c + \frac{2tc^m x^n}{(\lambda+1)} \left|\frac{\partial c}{\partial x}\right|^p \frac{\partial c}{\partial x} = f(t) , \tag{4.18}$$

and because of (4.17), $f(t)$ turns out to be at most a constant. Further from (4.15) and (4.17) we find that the appropriate value of λ is given by

$$\lambda = \left(\frac{l(m+1) + n - m + p(l-2)}{m + 2 - l(m+1) - n - p(l-2)} \right) , \tag{4.19}$$

which for p zero coincides with the value given in Hill [6], also arising from the above equations (4.3) and (4.6).

5. Integrals for high-order nonlinear evolution equations

In a recent paper [7] the authors have deduced integrals for similarity solutions of a general class of high-order nonlinear evolution equations represented by

$$\frac{\partial c}{\partial t} = \sum_{j=1}^{n} \alpha_j \frac{\partial^j}{\partial x^j} (c^{m_j}) , \tag{5.1}$$

for certain real constants m_j and α_j denote any real constants. In this section we show that the results derived in [7] can also be generated by the procedure described here. Following the notation used in [7], equation (5.1) admits similarity solutions of the form

$$c(x, t) = x^{-s}\phi(x/t^q) , \tag{5.2}$$

for real constants s and q provided that each m_j is defined by

$$m_j = 1 + (q^{-1} - j)/s , \tag{5.3}$$

which we assume to be the case for all j for which α_j is non-zero. An appropriate one-parameter group leaving (5.1) invariant is

$$x_1 = e^{-\varepsilon/s}x , \qquad t_1 = e^{-\varepsilon/sq}t , \qquad c_1 = e^{\varepsilon}c , \tag{5.4}$$

with corresponding first order partial differential equation

$$x \frac{\partial c}{\partial x} + sc + \frac{t}{q} \frac{\partial c}{\partial t} = 0 . \tag{5.5}$$

The integrals derived in Hill and Hill [7] can be obtained by examination of the equation resulting from (5.1) and (5.5) by elimination of $\partial c/\partial t$. For example, suppose $\alpha_j = 0$ for $j = 1, 2, \ldots, k - 1 (k < n)$ then we have

$$x \frac{\partial c}{\partial x} + sc + \frac{t}{q} \sum_{j=k}^{n} \alpha_j \frac{\partial^j(c^{m_j})}{\partial x^j} = 0 , \tag{5.6}$$

and on multiplication by x^{s-1} we see that this equation can be integrated with respect to x for $s = 1, 2, \ldots, k$ because terms involved in the summation can be eventually integrated by repeated integration by parts provided $s - 1$ is zero or a positive integer and $j - (s-1) \geq 1$ or in other words s much be less than or equal to each j. Thus integrals exist for multi-pole solutions with $s = 1, 2, \ldots, k$ and these results are detailed in [7].

Here we merely illustrate the process with reference to the single term equation

$$\frac{\partial c}{\partial t} = -\frac{\partial^3}{\partial x^3}(c^m), \tag{5.7}$$

which represents a nonlinear wave phenomena which is dominated by dispersion for which we have

$$q = [3 + (m-1)s]^{-1}. \tag{5.8}$$

From (5.5) and (5.7) we obtain

$$x\frac{\partial c}{\partial x} + sc = \frac{t}{q}\frac{\partial^3}{\partial x^3}(c^m), \tag{5.9}$$

which for $s = 1$ integrates immediately to give

$$xc - (m+2)t\frac{\partial^2}{\partial x^2}(c^m) = f_1(t), \tag{5.10}$$

and we see from (5.4) that $f_1(t)$ is at most a constant. On multiplication of (5.9) by x we may deduce the integral for $s = 2$, thus

$$x^2c - (2m+1)t\left\{x\frac{\partial^2(c^m)}{\partial x^2} - \frac{\partial(c^m)}{\partial x}\right\} = f_2(t), \tag{5.11}$$

and $f_2(t)$ can be shown to be at most a constant. Similarly, on multiplying (5.9) by x^2 we can deduce by two integrations by parts the following integral valid for $s = 3$, namely

$$x^3c - 3mt\left\{x^2\frac{\partial^2(c^m)}{\partial x^2} - 2x\frac{\partial(c^m)}{\partial x} + 2c^m\right\} = f_3(t), \tag{5.12}$$

and again $f_3(t)$ turns out to be a constant. The integrals (5.10)–(5.12) can be shown to agree with those derived in [7] and we refer the reader to this paper for further details and for special exact solutions arising from the integrals for particular values of m.

Finally in this section we may confirm that the last integral valid for $s = 3$ also arises by a time integration as follows. We first write equation (5.7) in the form

$$\frac{\partial c}{\partial t} + m\frac{\partial^2}{\partial x^2}\left(c^{m-1}\frac{\partial c}{\partial x}\right) = 0, \tag{5.13}$$

so that on eliminating $\partial c/\partial x$ by means of (5.5) we may deduce

$$\frac{\partial c}{\partial t} - \frac{1}{q}\frac{\partial^2}{\partial x^2}\left\{\frac{1}{x}\left(t\frac{\partial(c^m)}{\partial t} + sqm\, c^m\right)\right\} = 0. \tag{5.14}$$

Clearly if sqm is unity, in which case from (5.8) we have $s = 3$, equation (5.14) permits an integration with respect to time and we have

$$c - 3mt\frac{\partial^2}{\partial x^2}\left(\frac{c^m}{x}\right) = g(x), \tag{5.15}$$

and (5.4) with $s = 3$ implies that $g(x) = Cx^{-3}$ where C denotes an arbitrary constant. This integral evidently agrees with equation (5.12).

6. New exact solution for nonlinear diffusion with $m = 1$

In this section we derive a new exact solution for nonlinear diffusion not included in the special exact solutions given in [4]. The integral arising from (2.3) and applying to $\lambda = -m/(m+2)$ becomes

$$\frac{\phi^m \phi'}{\xi} - \frac{2\phi^{m+1}}{(m+2)\xi^2} + \frac{2\phi}{(m+2)^2} = C_1 , \tag{6.1}$$

where the prime denotes differentiation with respect to ξ and the concentration in this case is given by

$$c(x, t) = \frac{\phi(\xi)}{x} , \qquad \xi = \frac{x^{(m+2)/2}}{t^{1/2}} . \tag{6.2}$$

On making the change of variables

$$\phi(\xi) = \eta\psi(\eta) , \qquad \eta = \xi^{2/(m+2)} , \tag{6.3}$$

it is not difficult to show that equation (6.1) becomes

$$\psi^m \frac{d\psi}{d\eta} + \frac{\eta\psi}{(m+2)} = C , \tag{6.4}$$

where the new constant $C = (m+2)C_1/2$. It happens that for $m = 1$, this equation can be solved by the following successive substitutions,

$$\psi = V - \frac{\eta^2}{6} , \qquad \eta(V) = \frac{6C}{\Psi(V)} \frac{d\Psi}{dV} , \qquad V = (6C^2)^{1/3}\rho , \tag{6.5}$$

which give the Airy equation

$$\frac{d^2\Psi}{d\rho^2} - \rho\Psi = 0 , \tag{6.6}$$

which has the solution

$$\Psi(\rho) = C_1^* \rho^{1/2} I_{1/3}\left(\frac{2}{3}\rho^{3/2}\right) + C_2^* \rho^{1/2} I_{-1/3}\left(\frac{2}{3}\rho^{3/2}\right) , \tag{6.7}$$

where C_1^* and C_2^* denote arbitrary constants and $I_{1/3}$ and $I_{-1/3}$ denote the usual modified Bessel functions. From the equation

$$\eta(\rho) = \frac{(6^2C)^{1/3}}{\Psi(\rho)} \frac{d\Psi}{d\rho} , \tag{6.8}$$

we may deduce

$$\eta(\rho) = (6^2 C)^{1/3}\left\{\frac{1}{2\rho} + \rho^{1/2}\frac{[I'_{1/3}(\frac{2}{3}\rho^{3/2}) + C^* I'_{-1/3}(\frac{2}{3}\rho^{3/2})]}{[I_{1/3}(\frac{2}{3}\rho^{3/2}) + C^* I_{-1/3}(\frac{2}{3}\rho^{3/2})]}\right\}, \quad (6.9)$$

where the constant C^* denotes C_2^*/C_1^* and primes here denote differentiation with respect to the indicated argument.

Thus, altogether the general solution for $m = 1$, involving two arbitrary constants C and C^* is given by

$$c(x, t) = \frac{\psi(\eta)}{t^{1/3}}, \qquad \eta = \frac{x}{t^{1/3}}, \quad (6.10)$$

where the function ψ and η have parametric representations in terms of the parameter ρ as follows, namely

$$\psi(\rho) = (6C^2)^{1/3}\rho - \frac{\eta(\rho)^2}{6}, \quad (6.11)$$

and $\eta(\rho)$ is defined explicitly by equation (6.9).

Figures 1 and 2 show concentration curves for two values of the constant C^* and for $C = 1/3$ in both cases. In Fig. 1 the constant C^* has the value infinity so that η is zero when

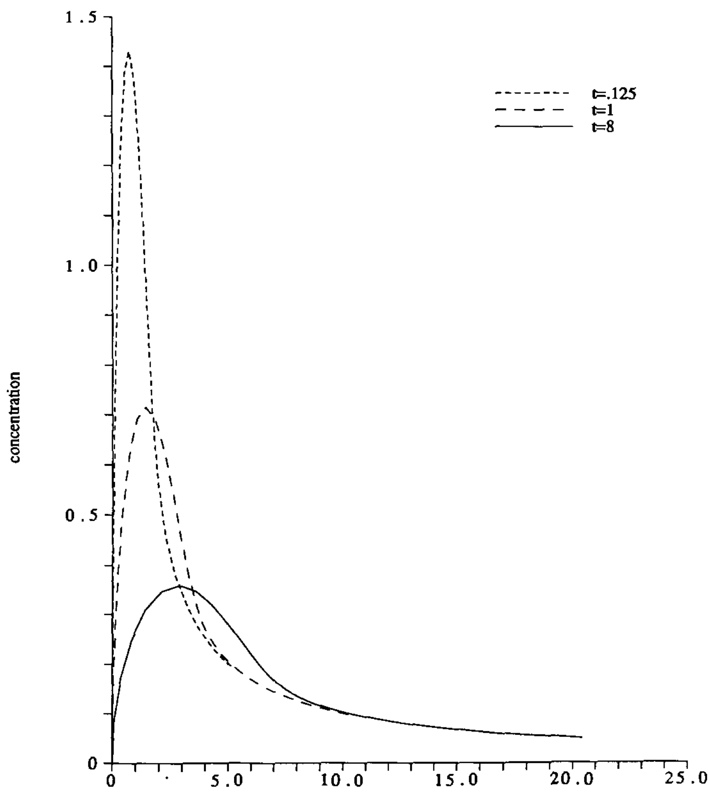

Fig. 1. Variation of concentration with position for three times, calculated from (6.9)–(6.11) with C^* infinite and $C = 1/3$.

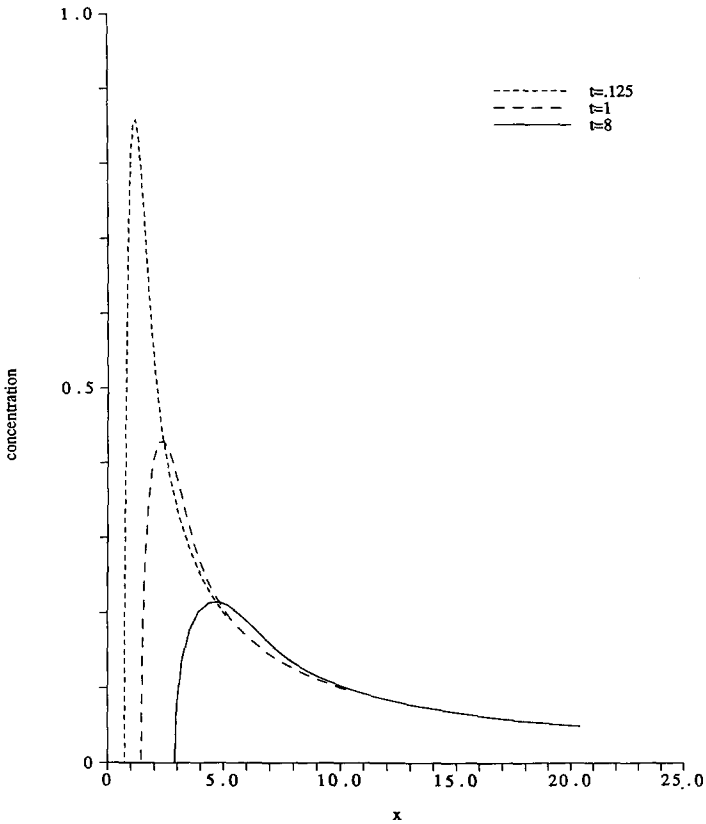

Fig. 2. Variation of concentration with position for three times, calculated from (6.9)–(6.11) with $C^* = 1$ and $C = 1/3$.

ρ is zero and therefore $c(0, t)$ vanishes for all time and accordingly there is a fixed boundary at the origin. For other values of C^* giving a physically sensible concentration (that is, positive) there is a moving boundary that starts at the origin at time $t = 0$ and subsequently moves to the right and Fig. 2 shows this behaviour for $C^* = 1$. Different values of the constant C produce very little variation in the concentration curves. There are certainly no qualitative changes and only small quantitative changes even when the constant C is negative.

References

1. C. Atkinson and C.W. Jones, Similarity solutions in some non-linear diffusion problems and in boundary-layer flow of a pseudo-plastic fluid. *Q. Jl. Mech. Appl. Math.* 27 (1974) 193–211.
2. R.E. Grundy, Similarity solutions of the nonlinear diffusion equation. *Quart. Appl. Math.* 37 (1979) 259–280.
3. R.E. Grundy, A mathematical model for rock blasting involving a degenerate nonlinear diffusion equation. *Q. Jl. Mech. Appl. Math.* 43 (1990) 173–188.
4. D.L. Hill and J.M. Hill, Similarity solutions for nonlinear diffusion – further exact results. *J. Engng. Math.* 24 (1990) 109–124.
5. D.L. Hill and J.M. Hill, Utilization of nonlinear diffusion solutions for n-diffusion and power-law materials. *Acta Mechanica* 82 (1991).
6. J.M. Hill, Similarity solutions for nonlinear diffusion – a new integration procedure. *J. Engng. Math.* 23 (1989) 141–155.
7. J.M. Hill and D.L. Hill, High-order nonlinear evolution equations. *IMA J. Appl. Math.* 45 (1990) 243–265.
8. J.R. King, Exact solutions to some nonlinear diffusion equations. *Q. Jl. Mech. Appl. Math.* 42 (1989) 537–552.